GOSFORD HIGH SCHOOL

2008 HIGHER SCHOOL CERTIFICATE

MATHEMATICS EXTENSION 2

ASSESSMENT TASK #1

DECEMBER 2007

Time Allowed – 105 minutes + 5 minutes reading time

All necessary working should be shown.

Full marks may not be awarded for unnecessarily untidy work or work that is poorly organized.

Students must begin each new question on a new page.

Students need to place their name and/or HSC candidate number at the top of each new page.

Questions will be collected separately at the conclusion of the assessment task.

All questions are to be attempted.

Extension 2 Mathematics Assessment Term 4 2007

Question 1 (17 marks) **Marks**

(a) Evaluate i^{2007} **1**

(b) Let $z = 3 - 2i$ and $w = 3 + 4i$.

Find, in the form $x + iy$.

(i) $\bar{z}$ **1**

(ii) $z\bar{z}$ **1**

(iii) $\dfrac{z}{w}$ **1**

(c) (i) Express $1 + \sqrt{3}i$ in the modulus argument form. **2**

(ii) Express $\left(1 + \sqrt{3}i\right)^7$ in the modulus argument form. **2**

(iii) Express $\left(1 + \sqrt{3}i\right)^7$ in the form $x + iy$. **1**

(d) Sketch on separate Argand diagrams

(i) $|z - i| = |z + 1|$ **2**

(ii) $|z - 3| = 3$ **2**

(iii) $Arg(z - 1) = \dfrac{3\pi}{4}$ **2**

(e) Mark clearly on the Argand diagram the region satisfied by **2**

simultaneously by $-1 \le \text{Im}(z) \le 1$ and $-\dfrac{\pi}{4} \le Arg(z) \le \dfrac{\pi}{4}$.

15

Question 2 (18 marks) **Marks**

(a) Find the square root of $8+6i$ 3

(b) Find all the solutions of the equation $z^5 = 1$. 3

Give your answers in modulus-argument form.

(c) If ω is a root of $z^3 - 1 = 0$, show that

(i) $\omega^3 = 1$ 1

(ii) $1+\omega+\omega^2 = 0$ 1

(d) Using the results from question 2 (c) or otherwise 2

evaluate $\dfrac{1}{1+\omega}+\dfrac{1}{1+\omega^2}$

(e) The diagram below shows the locus of points z in the complex plane such that 4

$\arg(z-3)-\arg(z+1)=\dfrac{\pi}{3}$.

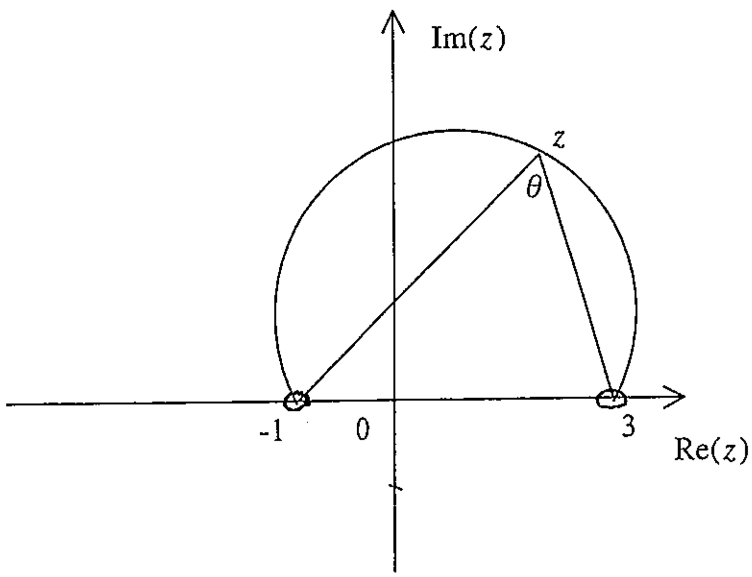
This locus is part of a circle.

The angle between the lines from -1 to z and from 3 to z is θ, as shown.

Copy this diagram onto your answer sheet.

(i) Explain why $\theta = \dfrac{\pi}{3}$. 2

(ii) Find the centre of the circle. 3



Question 3 (~~15~~ 20 marks) **Marks**

(a) By considering the first derivative of $y = \{f(x)\}^n$, where is a positive integer greater than, show that the stationary points on the curve $y = \{f(x)\}^n$ occur at the same x values as the stationary points on the original curve and where the curve cuts the x axis. **2**

(b) Given the sketch of $y = f(x)$. Draw neat sketches of

(i) $y = f(-x)$ 1

(ii) $y = |f(x)|$ 1

(iii) $y = f(|x|)$ 1

(iv) $y = \{f(x)\}^2$ **2**

(v) $y = \sqrt{f(x)}$ 3 "Linear"

(vi) $y = \dfrac{1}{f(x)}$ **2**

(c) Draw **on the same number plane** a neat sketch of

(i) $y = x^2(x-1)$ **1**

and (ii) $y^2 = x^2(x-1)$ 3

(d) Evaluate $\lim\limits_{n \to \infty} 4 - \dfrac{n+2}{2^{n-1}}$ **1**

Question 4 (18 marks) **Marks**

(a) Sketch the graph of $f(x) = x^4 - 4x^3 + 4x^2$. **3**

Hence show that $x^4 - 4x^3 + 4x^2 - \frac{1}{2} = 0$ has four real roots.

(b) Find the equation of the tangent to the curve **3**

$x^3 - 3xy + y^3 = 13$ at the point $(2, -1)$.

(c) (i) Show that $\frac{x^2 + 3x}{x - 1} = x + 4 + \frac{4}{x - 1}$. **1**

(ii) Write down the equation of any asymptotes of **2**

$y = \frac{x^2 + 3x}{x - 1}$.

(iii) Find the coordinates of the points of intersection **1**

of the curve with the x axis.

(iv) Draw a neat sketch of the curve. **2**

(d) Given the complex number $z = \cos\theta + i\sin\theta$ show that

$z^n + z^{-n} = 2\cos n\theta$ **2**

(e) O is the origin and A is the point represented by the complex **2**

number $2 - i\sqrt{5}$. A right angled triangle is formed by rotating OA anticlockwise through 90°. Find the length of the hypotenuse of this triangle.

(f) The complex number $\frac{z - 1}{z - 2i}$, where $z = x + iy$, is real. Show **2**

that the locus of z is $2x + y = 2$.

Question 1.

a) $i^{2007} = i^3$

$= i(i^2)$

$= i \times -1$

$= -i$

b)(i) $\bar{z} = \overline{3-2i}$

$= 3+2i$

(ii) $z\bar{z} = (3-2i)(3+2i)$

$= 9-4i^2$

$= 9+4$

$= 13$

(iii) $\dfrac{z}{w} = \dfrac{3-2i}{3+4i}$

$= \dfrac{3-2i}{3+4i} \times \dfrac{3-4i}{3-4i}$

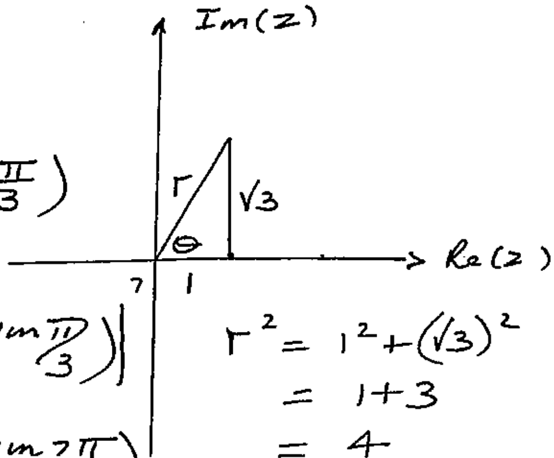
$= \dfrac{9-12i-6i+8i^2}{9-16i^2}$

$= \dfrac{9-18i-8}{9+16}$

$= \dfrac{1-18i}{25}$

$= \dfrac{1}{25} - \dfrac{18}{25} i$

c) (i) $1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$



$r^2 = 1^2 + (\sqrt{3})^2$

$= 1+3$

$= 4$

$r = 2.$

$\tan\theta = \frac{\sqrt{3}}{1}$

$\theta = \frac{\pi}{3}.$

(ii) $(1+\sqrt{3}i)^7 = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^7$

$= 2^7\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)$

$= 2^7\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(iii) $(1+\sqrt{3}i)^7 = 2^7\left(\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right)$

$= 2^6(1+i\sqrt{3})$

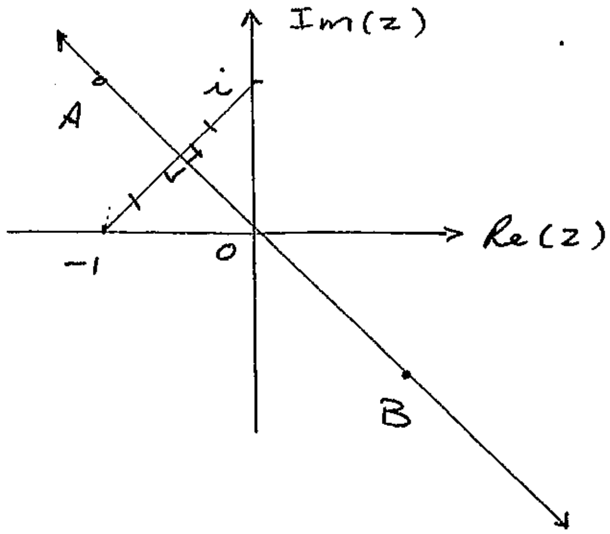
$= 64 + 64\sqrt{3}\, i$

d) (i) $|z-i| = |z+1|$

Locus is the line AB

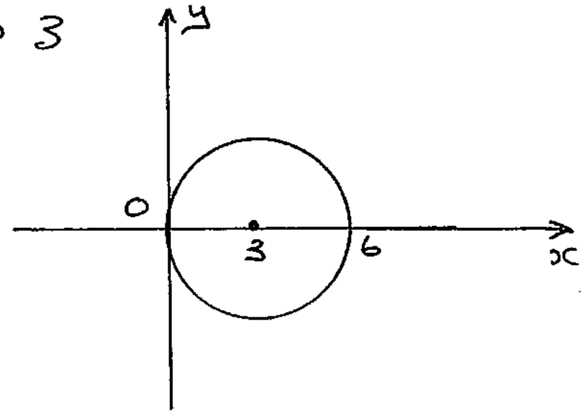
i.e. the $\perp$ bisector

of $(-1,0)$ and $(0,1)$

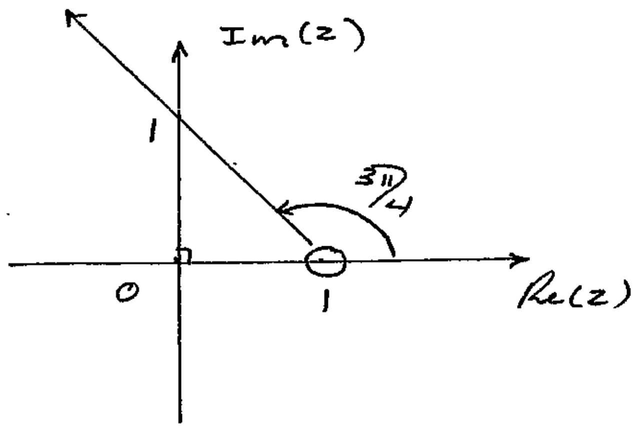


(ii) $|z-3| = 3$.

Circle centre $(3,0)$ Radius 3

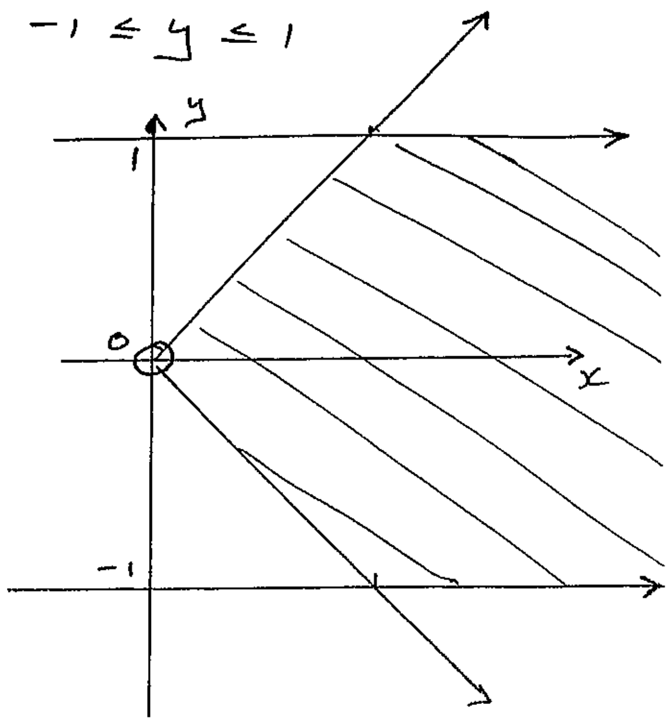


(iii) $\text{Arg}(z-1) = 3\pi/4$



e) $-1 \leq \text{Im}(z) \leq 1 \Rightarrow -1 \leq y \leq 1$

$-\pi/4 \leq \text{Arg}\, z \leq \pi/4$



Question 2.

a) let $x + iy = \sqrt{8+6i}$ where x & y are real

$$(x+iy)^2 = 8+6i$$

$$x^2 + 2ixy + i^2y^2 = 8+6i$$

$$x^2 - y^2 + 2ixy = 8+6i$$

$$x^2 - y^2 = 8 \qquad 2xy = 6$$

$$y = \frac{6}{2x}$$

$$y = \frac{3}{x}$$

Substitute $\frac{3}{x}$ for y in $x^2 - y^2 = 8$

$$\therefore \quad x^2 - \left(\frac{3}{x}\right)^2 = 8$$

$$x^2 - \frac{9}{x^2} = 8$$

$$x^4 - 9 = 8x^2$$

$$x^4 - 8x^2 - 9 = 0$$

$$(x^2-9)(x^2+1) = 0$$

$x^2 - 9 = 0$ OR $x^2 + 1 = 0$

$x^2 = 9$ $\qquad$ $x^2 = -1$

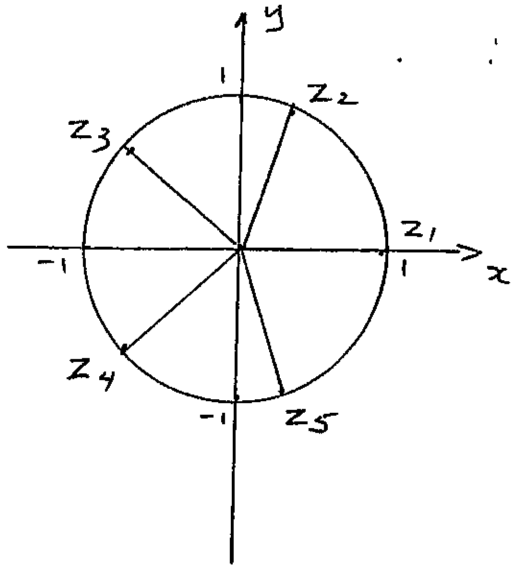
$x = \pm 3$ $\qquad$ No solutions since x is real.

If $x = 3$, $y = \frac{3}{3} = 1$ $\qquad$ If $x = -3$, $y = \frac{3}{-3} = -1$

$$\therefore \quad \sqrt{8+6i} = 3+i \text{ OR } -3-i$$

$$= \pm(3+i)$$

b) $z^5 = 1$



Roots are on the unit circle starting at $\theta = 0$ and spaced at $\angle$'s of $\frac{2\pi}{5}$.

$\therefore z_1 = 1(\cos 0 + i \sin 0)$

$z_2 = 1(\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5})$

$z_3 = 1(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5})$

$z_4 = 1(\cos 6\pi/5 + i \sin 6\pi/5)$

$z_5 = 1(\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5})$

c) $z^3 = 1$

(i) ω is a root of $z^3 = 1$ $\therefore$ ω satisfies $z^3 = 1$

i.e. $\omega^3 = 1$

(ii) $z^3 = 1$

$z^3 - 1 = 0$

$(z-1)(z^2+z+1) = 0$

$z - 1 = 0$ OR $z^2 + z + 1 = 0$

ω is a complex root

$\therefore$ it is a root of $z^2 + z + 1 = 0$

and satisfies $z^2 + z + 1 = 0$

$\omega^2 + \omega + 1 = 0.$

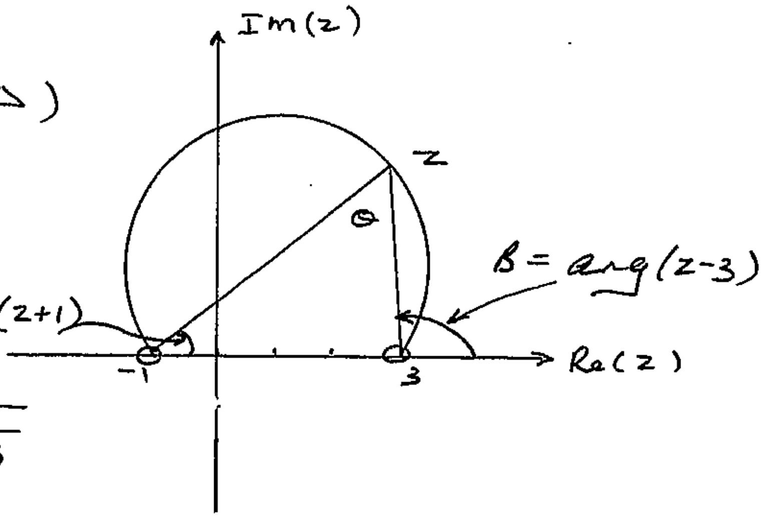
d) $\frac{1}{1+\omega} + \frac{1}{1+\omega^2} = \frac{1}{-\omega^2} + \frac{1}{-\omega}.$ since $1+\omega+\omega^2=0$

$= \frac{-\omega - \omega^2}{(-\omega^2)(-\omega)}$

$= \frac{-(\omega+\omega^2)}{\omega^3}$

$$= \frac{-(-1)}{1}$$
$$= 1.$$

e) (i)



$\alpha + \theta = \beta$ (ext. ∠ of △)

$\theta = \beta - \alpha$

$= \arg(z-3) - \arg(z+1)$

But

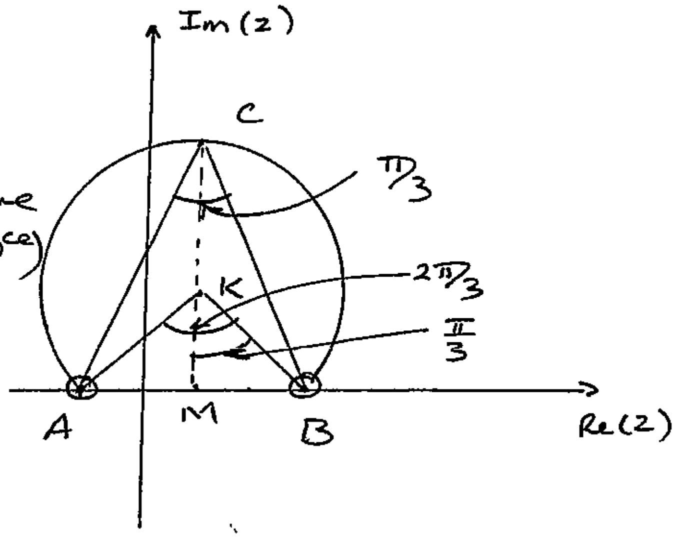
$\arg(z-3) - \arg(z+1) = \frac{\pi}{3}$

$\therefore \theta = \pi/3$

(ii) $\angle C = \pi/3$

$\angle AKB = 2\pi/3$ (∠ at centre = 2 × ∠ at ⊙ce)

$\therefore \angle MKB = \frac{\pi}{3}$



$\therefore \frac{MB}{MK} = \tan \pi/3$

$\frac{MB}{MK} = \sqrt{3}$

$\frac{2}{MK} = \sqrt{3}$

$\frac{2}{\sqrt{3}} = MK$

Centre is at $\left(2, \frac{2}{\sqrt{3}}\right)$

Question 3.

a) $y = (f(x))^n$

$$\frac{dy}{dx} = n(f(x))^{n-1} f'(x)$$

Stat. pts. occur when $\frac{dy}{dx} = 0$

i.e. $n(f(x))^{n-1} f'(x) = 0$

i.e. $(f(x))^{n-1} = 0$ OR $f'(x) = 0$

$f(x) = 0$ OR $f'(x) = 0$

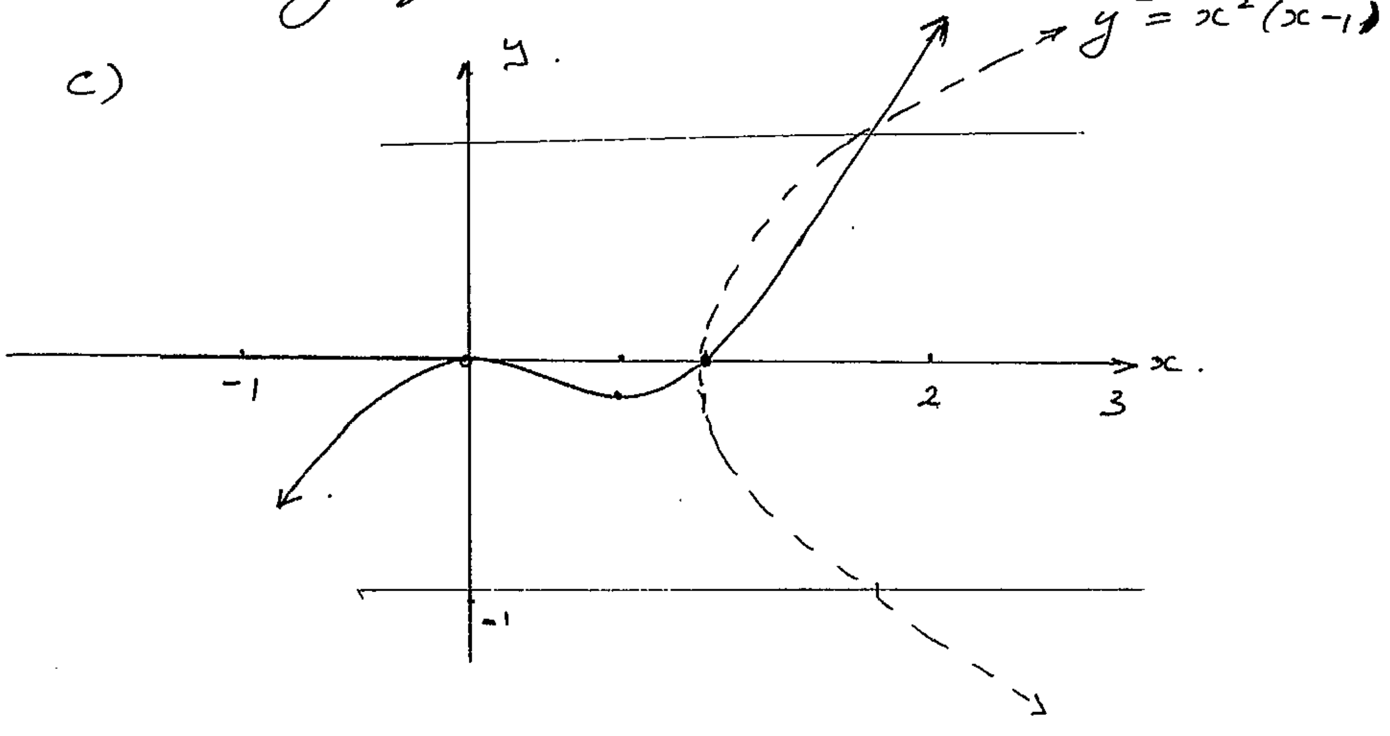
$f(x) = 0$ gives the x values when $f(x)$ cuts the x axis

$f'(x) = 0$ gives the x values of the stationary points.

Hence the result is true.

b) See graphs.

c)



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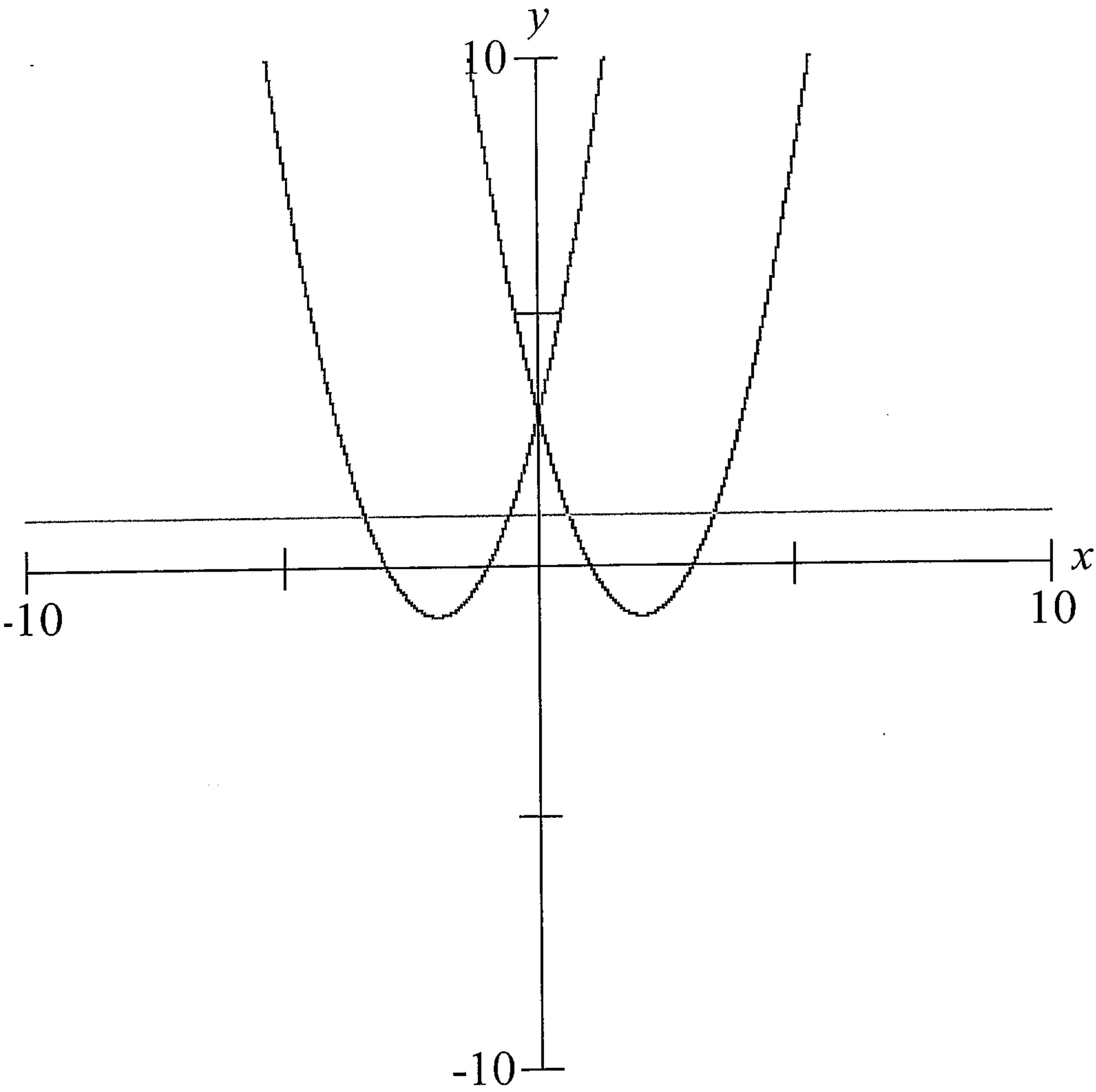
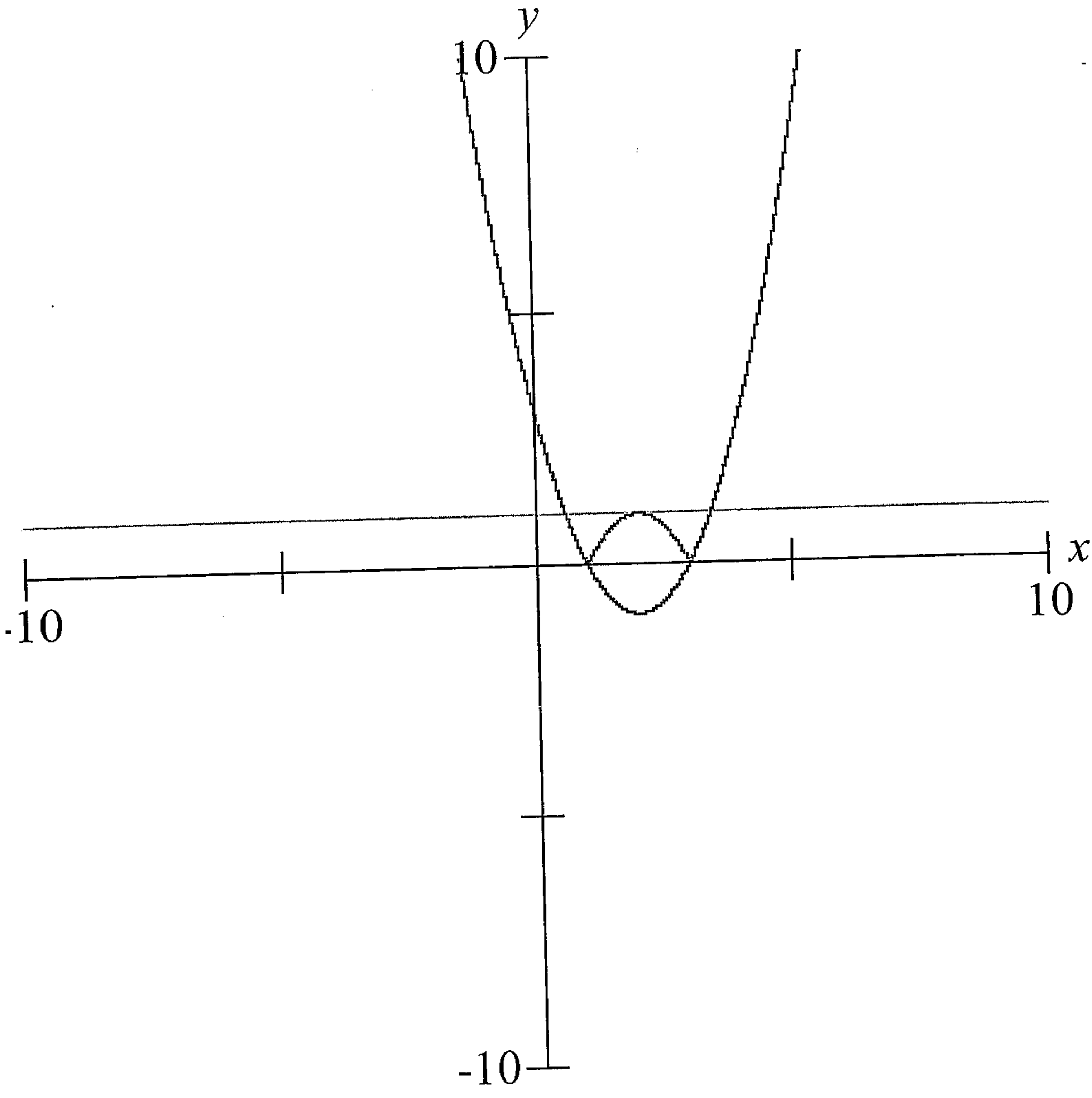


$y=(x-1)\cdot(x-3)$ $\quad$ $y=1$ $\quad$ $y=(-x-1)\cdot(-x-3)$



$y=(x-1)\cdot(x-3)$ $y=1$ $y=|(x-1)\cdot(x-3)|$

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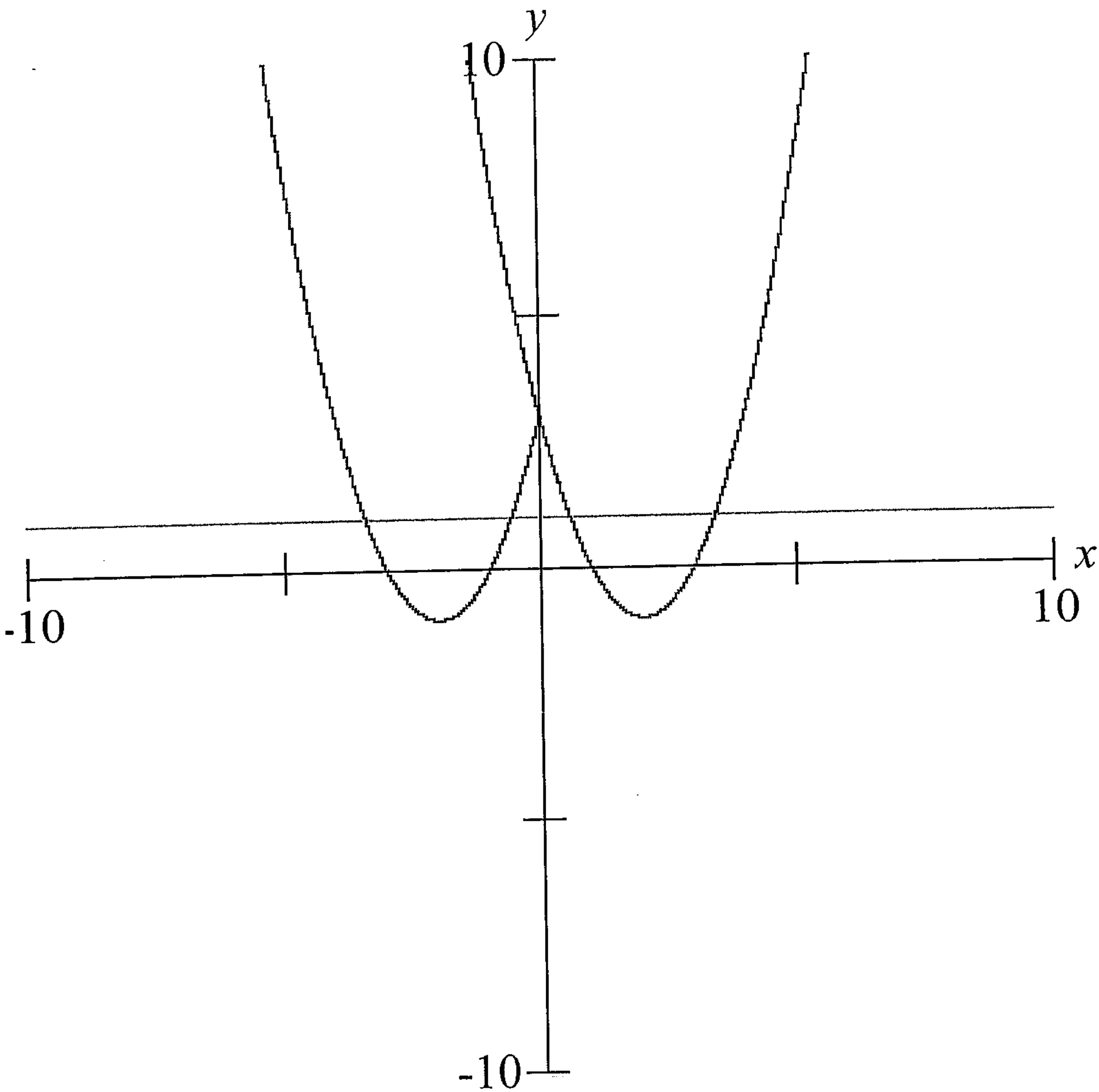
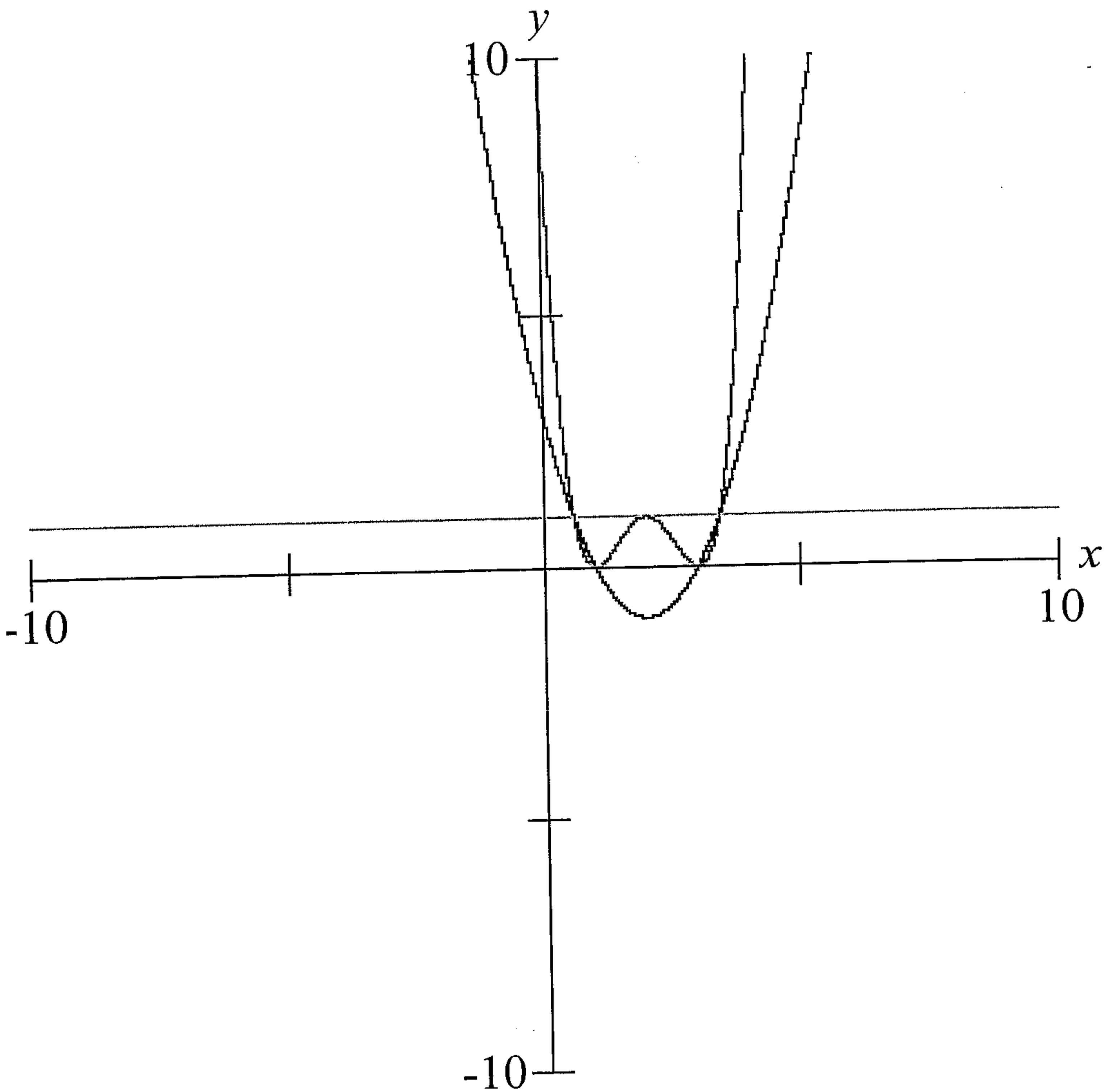
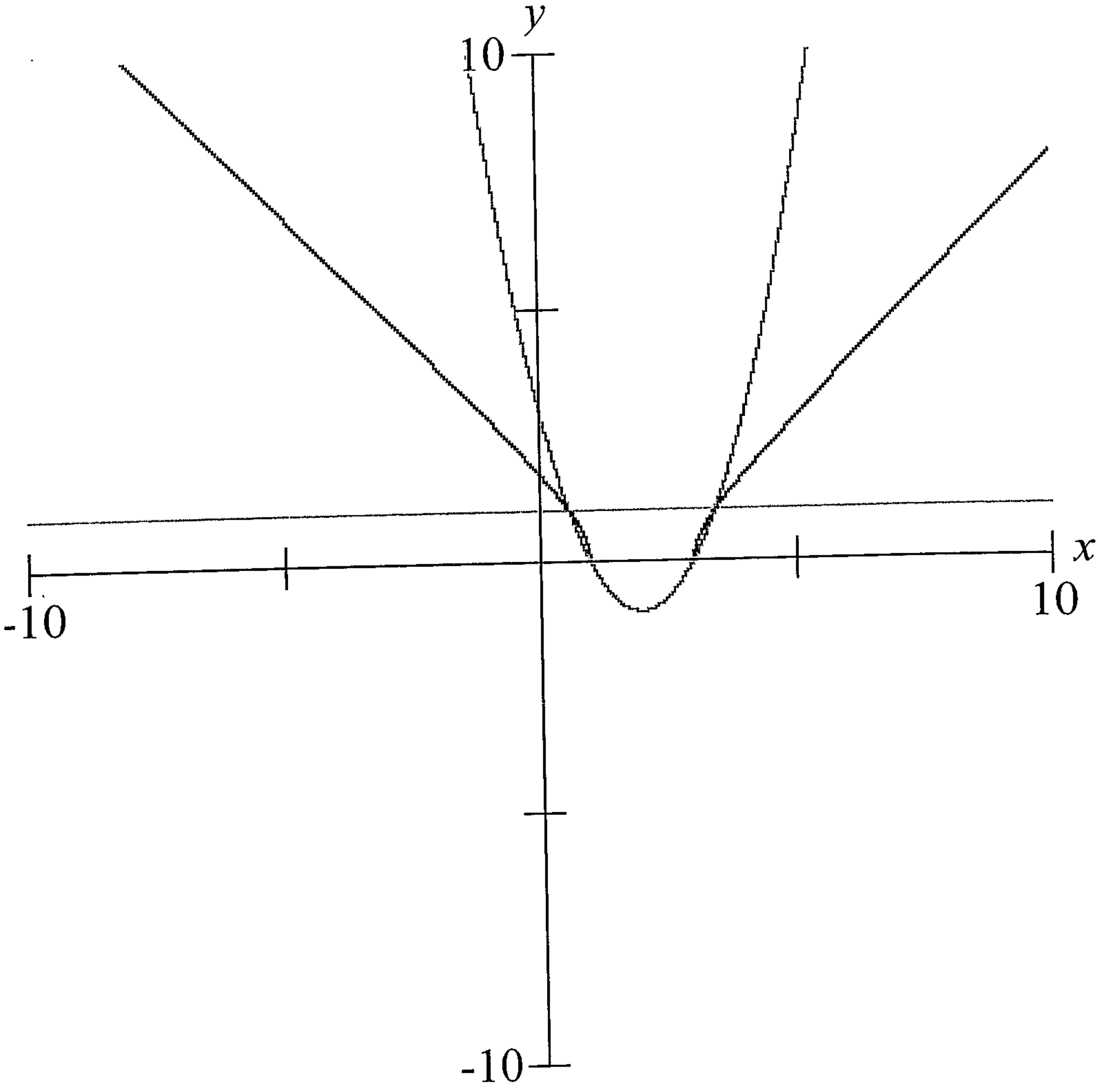


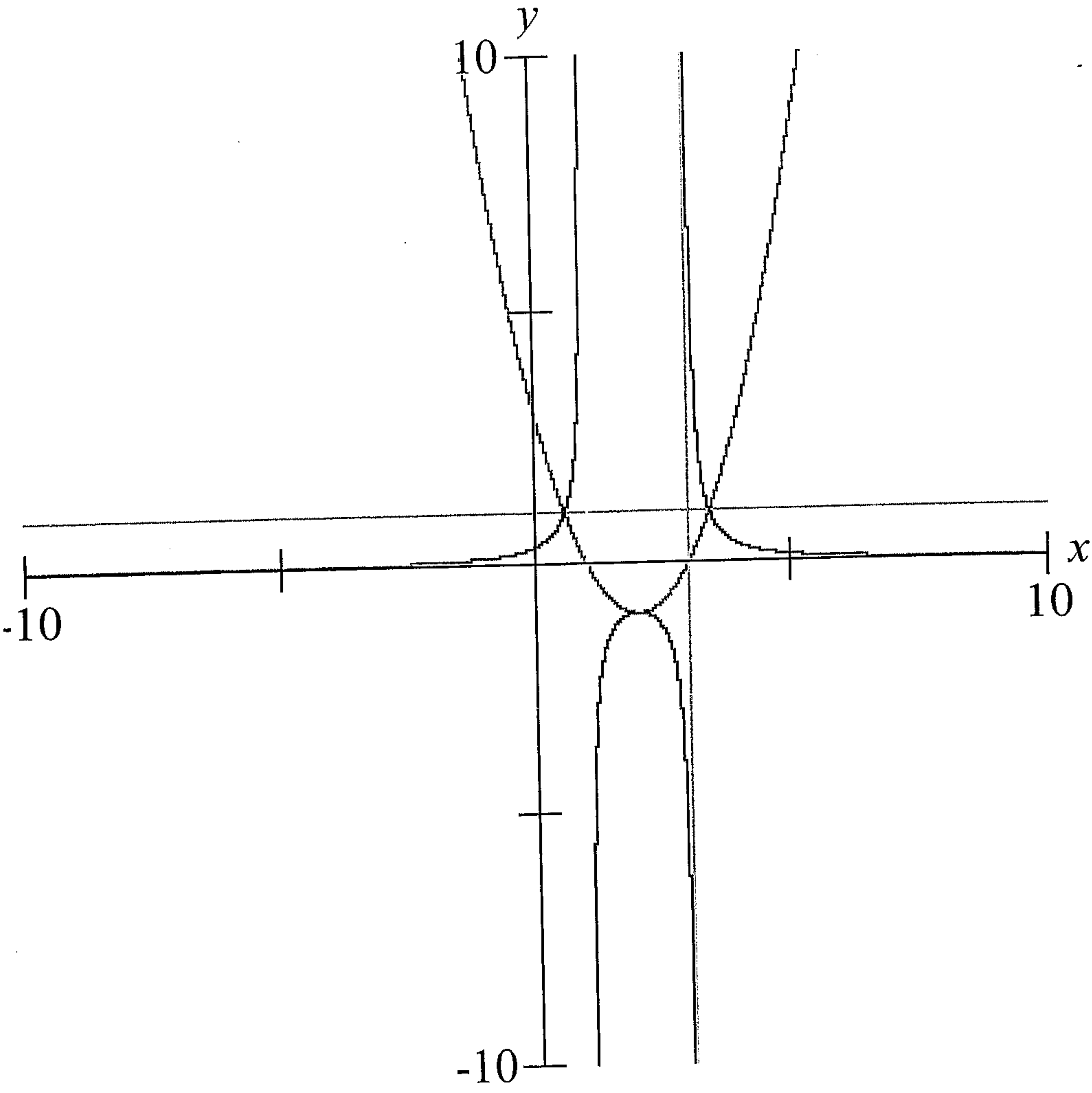
$y=(x-1)\cdot(x-3)$ $y=1$ $y=(|x|-1)\cdot(|x|-3)$



$y=(x-1)\cdot(x-3)$ $\quad$ $y=1$ $\quad$ $y=((x-1)\cdot(x-3))^2$



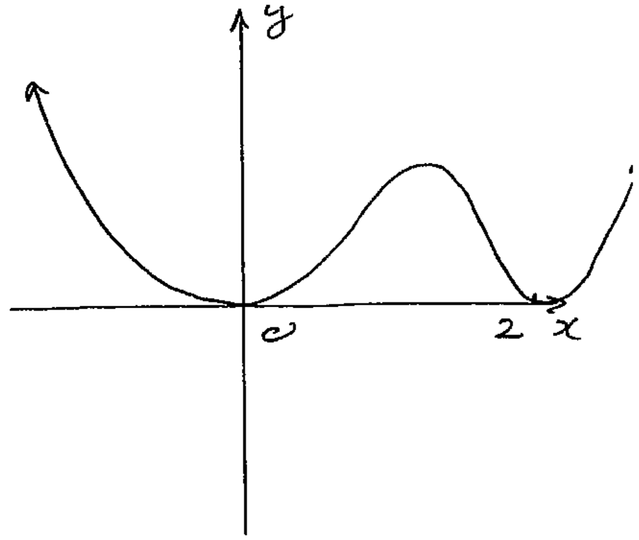
$y=1$ $\quad$ $y=\sqrt{(x-1)\cdot(x-3)}$ $\quad$ $y=(x-1)\cdot(x-3)$



$y=1$ $\qquad y=\frac{1}{((x-1)\cdot(x-3))}$ $\qquad y=(x-1)\cdot(x-3)$ $\qquad x=1$ $\qquad x=3$

Question 4

$$f(x) = x^4 - 4x^3 + 4x^2$$
$$= x^2(x^2 - 4x + 4)$$
$$= x^2(x-2)^2$$



$f(x) = 0$ has double roots at $x = 0$ & $x = 2$

Stat. pt. between $x = 0$ & $x = 2$

$$f'(x) = 4x^3 - 12x^2 + 8x$$
$$= 4x(x^2 - 3x + 2)$$
$$= 4x(x-1)(x-2)$$

Stat pts when $f'(x) = 0$

i.e. $x = 0$, $x = 1$, $x = 2$

at $x = 0$, $y = 0$

at $x = 2$, $y = 0$

at $x = 1$, $y = 1$

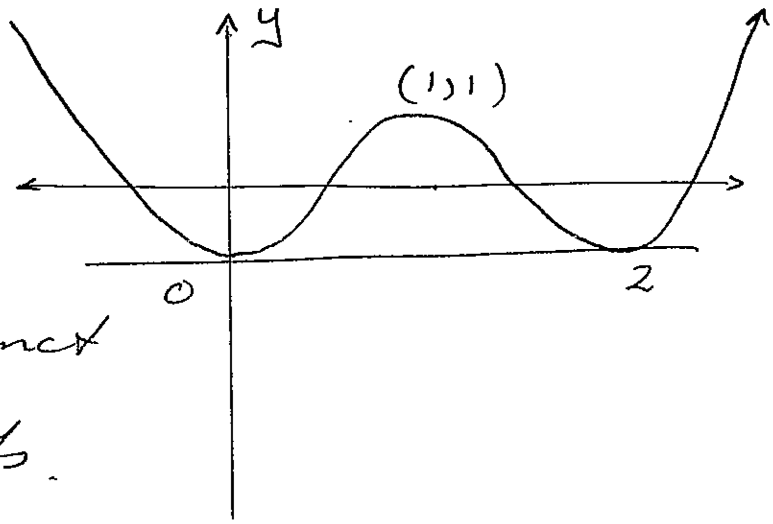
$\therefore$ Max turning point between $x = 0$ and $x = 2$ is at $(1, 1)$

$$x^4 - 4x^3 + 4x^2 - \frac{1}{2} = 0$$
$$x^4 - 4x^3 + 4x^2 = \frac{1}{2}$$

Draw $y = \frac{1}{2}$

Cuts $f(x)$ in 4 distinct points

$\therefore$ 4 real roots.



b) $x^3 - 3xy + y^2 = 13$

$$3x^2 - 3\left(x\frac{dy}{dx} + y\cdot 1\right) + 2y\frac{dy}{dx} = 0$$

$$3x^2 - 3x\frac{dy}{dx} - 3y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y-3) = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y-3x^2}{2y-3}$$

at $(2,-1)$, $\frac{dy}{dx} = \frac{3(-1) - 3\times 2^2}{2(-1) - 3}$

$$= \frac{-3-12}{-5}$$

$$= \frac{-15}{-5}$$

$$= 3.$$

$\therefore$ Equ of Tgt is

$$y - (-1) = 3(x-2)$$

$$y + 1 = 3x - 6$$

$$\underline{\underline{y = 3x - 7}}$$

c) (i) $x + 4 + \frac{4}{x-1} = \frac{(x+4)(x-1)+4}{x-1}$

$$= \frac{x^2 + 3x - 4 + 4}{x-1}$$

$$= \frac{x^2 + 3x}{x-1}$$

(ii) Vertical asymptote at $x - 1 = 0$
i.e. $x = 1$

As $x \to \infty$ $y \to x + 4$ $\therefore y = x + 4$ is an asymptote c)(iii) Cuts x axis when $y=0$

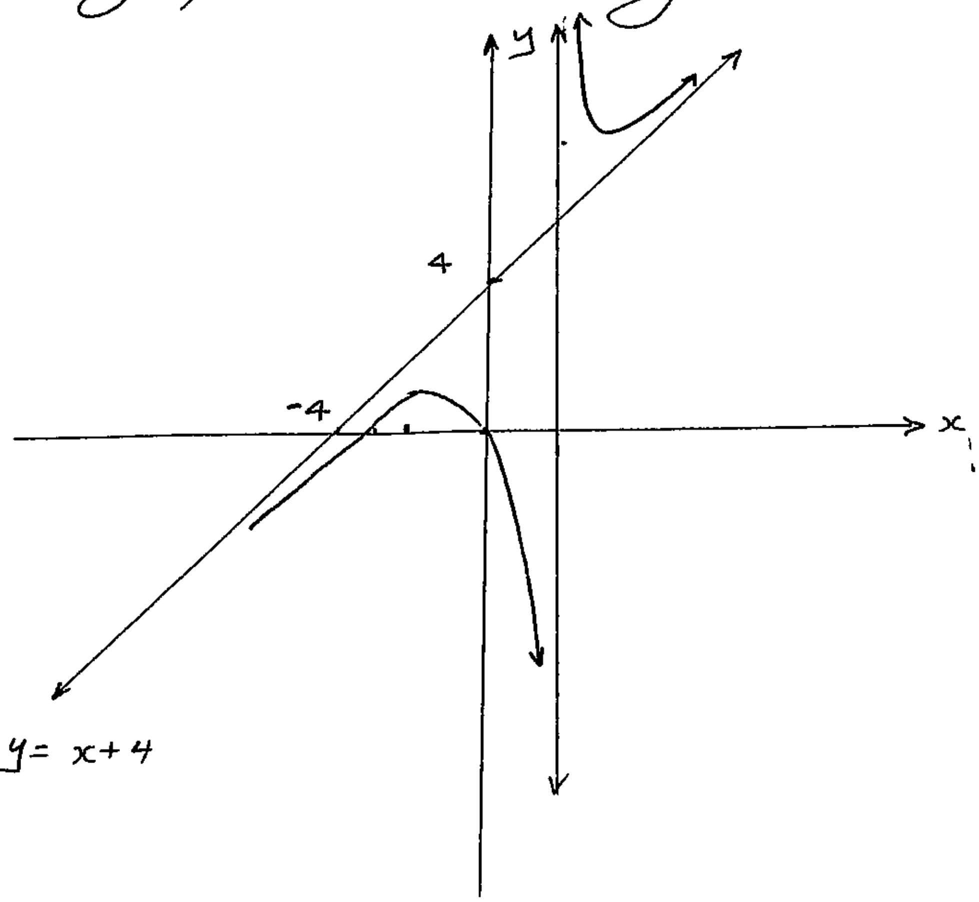
$$\therefore \frac{x^2+3x}{x-1} = 0.$$

$$x^2 + 3x = 0.$$

$$x(x+3) = 0$$

$$x = 0 \text{ OR } x = -3.$$

(iv) see graph (computer generated)



d) $z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin 2\theta$

$= 2\cos n\theta.$

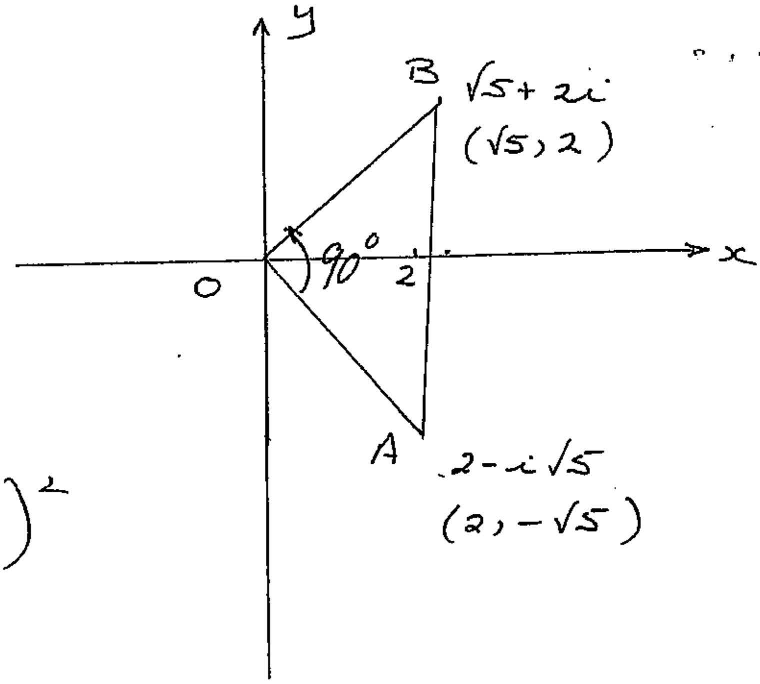
e) Rotate OA anticlockwise through $90°$

$\therefore$ the new point is $i(2 - i\sqrt{5})$

$= 2i - i^2\sqrt{5}$

$= +\sqrt{5} + 2i$

AB = hypotenuse.



$$AB^2 = |OA|^2 + |OB|^2$$

$$= 2\times |OB|^2$$

$$= 2\times\left(\sqrt{(\sqrt{5})^2+2^2}\right)^2$$

$$= 2\times\left|(\sqrt{5})^2+2^2\right|$$

$$= 2\times(5+4)$$

$$= 2\times 9$$

$$AB^2 = 18.$$

$$AB = \sqrt{18} = 3\sqrt{2} \text{ units}$$

f) $$\frac{z-1}{z-2i} = \frac{x+iy-1}{x+iy-2i}$$

$$= \frac{x-1+iy}{x+i(y-2)} \times \frac{x-i(y-2)}{x-i(y-2)}$$

$$= \frac{x^2-ix(y-2)-x+i(y-2)+ixy-i^2y(y-2)}{x^2-i^2(y-2)^2}$$

$$= \frac{x^2-x+y(y-2)}{x^2+(y-2)^2} + i\frac{[-x(y-2)+(y-2)+xy]}{x^2+(y-2)^2}$$

$$= \frac{x^2+y^2-x-2y}{x^2+(y-2)^2} + i\frac{(-xy+2x+y-2+xy)}{x^2+(y-2)^2}$$

$$= \frac{x^2+y^2-x-2y}{x^2+(y-2)^2} + i\frac{(2x+y-2)}{x^2+(y-2)^2}$$

Since $\frac{z-1}{z-2i}$ is real then $\text{Im}\left(\frac{z-1}{z-2i}\right)=0$.

$$\therefore \frac{2x+y-2}{x^2+(y-2)^2} = 0 \Rightarrow 2x+y-2=0$$

i.e. $2x+y-2=0$.